ON THE CHARACTERIZATION OF PERIODIC GENERALIZED HORADAM SEQUENCES

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Abstract. The Horadam sequence is a direct generalization of the Fibonacci numbers in the complex plane, which depends on a family of four complex parameters: two recurrence coefficients and two initial conditions. In this article a computational matrix-based method is developed to formulate necessary and sufficient conditions for the periodicity of generalized complex Horadam sequences, which are generated by higher-order recurrences for arbitrary initial conditions. The asymptotic behavior of generalized Horadam sequences generated by roots of unity is also examined, along with upper boundaries for the disk containing periodic orbits. Some applications are suggested, along with a number of future research directions.

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1. Introduction

Let $m \geq 2$ be a natural number, $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{c} = (c_1, \dots, c_m)$ be vectors of complex numbers and let $\{w_n(\mathbf{a}; \mathbf{c})\}_{n=0}^{\infty}$ be the sequence defined by the recurrence

$$w_n = c_1 w_{n-1} + c_2 w_{n-2} + \cdots + c_m w_{n-m}, \quad m \leq n \in \mathbb{N}, \tag{1.1}$$

satisfying the initial conditions $w_{i-1} = a_i, i = 1, \dots, m$.

Following the work of A.F. Horadam—who initiated the investigation of second order general recurrences of complex numbers in his 1965 paper [6]—the sequence arising from (1.1) is called a generalized Horadam sequence which contains many well known sequences as special cases, for particular choices of vectors $\mathbf{a}$ and $\mathbf{c}$. For example, one obtains the Fibonacci sequence for $(a_1, a_2) = (0, 1)$ and $(c_1, c_2) = (1, 1)$, and the Lucas sequence for $(a_1, a_2) = (0, 1)$ and $(c_1, c_2) = (1, -1)$. A historical perspective on results related to Horadam sequences is given in the survey paper of Larcombe *et al.* [8].

Under certain conditions, linear recurrent sequences can be periodic. Sufficient conditions for periodicity are formulated in [13] for generalized Lucas sequences over an associative ring with identity, or in [14] for arbitrary sequences over algebraic number fields. A comprehensive list of periodic recurrent sequences is detailed in [5, Chapter 3], with an emphasis on sequences defined over finite fields. The periodic Horadam sequences were characterized in [1], while the orbits with a given periodicity were enumerated in [3] (this result led to a new integer sequence, for which upper and lower boundaries have been determined).

In this paper are established necessary and sufficient conditions for the periodicity of generalized complex Horadam sequences. The results are derived using the formulas for the general term of arbitrary linear recurrences and are formulated in terms of the initial conditions $a_1, \dots, a_m$ and the generators $z_1, \dots, z_m$, representing the non-zero roots (distinct or equal) of the characteristic equation of (1.1)

$$\lambda^n = c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_{m-1} \lambda^{n-m+1} + c_m \lambda^{n-m}, \quad n \in \mathbb{N}. \tag{1.2}$$

The structure of the solution space for the recurrence (1.1) is first discussed, along with formulas for the general term of the sequence $\{w_n(\mathbf{a};\mathbf{c})\}_{n=0}^{\infty}$. This part is based on the work of Ivanov [7], which here is summarized and complemented by the numerical computation of special matrices, built from elementary blocks (Matlab® codes are provided in the Appendix).

The necessary and sufficient conditions for the periodicity of sequences $\{w_n(\mathbf{a};\mathbf{c})\}_{n=0}^{\infty}$ are then presented for the scenarios when the non-zero roots of (1.2) are all distinct, all equal, or in general, where the distinct roots have arbitrary multiplicities.

When the non-zero solutions of (1.2) are kth roots of unity, the asymptotic behaviour of subsequences $\{w_{Nk+j}(\mathbf{a};\mathbf{c})\}_{N=0}^{\infty}$, $j = 0,\dots,k-1$ is examined. The subsequences converge when the roots' multiplicities are all less than one and diverge otherwise. When divergent, the terms of the subsequences can be collinear if the maximum root multiplicity is two, and approach an asymptote for higher multiplicities. In the end, outer boundaries for the region containing the periodic orbits of $\{w_n(\mathbf{a};\mathbf{c})\}_{n=0}^{\infty}$ are derived, based on the triangle inequality.

2. Theoretical background

In this section we summarize the behaviour of the sequence $\{z^n\}_{n=0}^{\infty}$ for an arbitrary value of $z \in \mathbb{C}$, as presented in [1]. The structure of the solution space for general linear recurrences under arbitrary conditions is also discussed, based on the work of Ivanov [7].

2.1. The behaviour of sequence $\{z^n\}_{n=0}^{\infty}$. As shown later in this section, the sequence $\{z^n\}_{n=0}^{\infty}$ represents the building block for the solutions of linear recurrent sequences of complex numbers, and its behaviour is described in the following result [1, Lemma 2.1]:

Lemma 1. *Let $z = r\exp(2\pi i x) \in \mathbb{C}$ be a complex number, defined for $r > 0$ and $x \in \mathbb{R}$. The orbit of $\{z^n\}_{n=0}^{\infty}$ is*

(a) *a regular k-gon if $r = 1$ and $x = j/k \in \mathbb{Q}$, with $\gcd(j,k) = 1$;*
(b) *a dense subset of the unit circle for $r = 1$ and $x \in \mathbb{R} \setminus \mathbb{Q}$;*
(c) *an inward spiral for $r < 1$;*
(d) *an outward spiral for $r > 1$.*

2.2. The structure of the solution space for linear recurrences. A solution of the recurrence (1.1) is any function $w : \mathbb{N} \to \mathbb{C}$ satisfying the condition

$$w(n) = c_1 w(n-1) + c_2 w(n-2) + \cdots + c_m w(n-m), \quad n \geq m. \tag{2.1}$$

(The notation $w(n)$ is only used in this section, while for the rest of the paper the more compact subscript notation w_n is preferred). As each solution is completely determined by m initial conditions $w(0), w(1), \dots, w(m-1)$, the set containing the solutions of (2.1) forms a vector space V of dimension m over $\mathbb{C}$. The general term of the sequence satisfying the recurrence (2.1) is then a linear combination of m functions which form a basis for V.

Note that for a non-zero value of λ, the characteristic equation (1.2) is equivalent to

$$\lambda^m = c_1\lambda^{m-1} + c_2\lambda^{m-2} + \cdots + c_{m-1}\lambda + c_m. \tag{2.2}$$

It may be assumed without loss of generality that the order of the recurrence can not be reduced, therefore $c_m \neq 0$. For finding a base of the vector space V, one may first check that the functions $w(n) = \lambda^n$ ($\lambda \neq 0$) are a solution of (2.1), whenever λ is a root of the *characteristic polynomial*

$$f(x) = x^m - c_1 x^{m-1} - c_2 x^{m-2} - \cdots - c_{m-1}x - c_m. \tag{2.3}$$

As a complex polynomial, $f(x)$ has exactly m roots. Examples of bases for V when the roots of (2.3) are all distinct, all equal, or distinct with arbitrary multiplicities are presented below.

Theorem 2.1. *If the characteristic polynomial $f(x)$ defined in (2.3) has m distinct roots $z_1, \dots, z_m$, then the m sequences*

$$f_1(n) = z_1^n, f_2(n) = z_2^n, \dots, f_m(n) = z_m^n, \tag{2.4}$$

form a basis of the vector space V containing the solutions of the recurrence (2.1).

The proof is based on two facts. First, each function $f_i(n)$ is a solution of (2.1). Second, the Vandermonde determinant involving the first m values $1, z_i, \dots, z_i^{m-1}$ of each function $f_i(n)$ is non-zero for distinct values $z_1, \dots, z_m$. A detailed proof is presented in [7, Theorem 1].

Theorem 2.2. *If the characteristic polynomial $f(x)$ defined in (2.3) has m roots equal to z, then the m sequences*

$$f_1(n) = z^n, f_2(n) = nz^n, \dots, f_m(n) = n^{m-1}z^n, \tag{2.5}$$

form a basis of the vector space V containing the solutions of the recurrence (2.1).

The main idea of the argument is that a multiple root of polynomial (2.3) is also a root of its derivative. A detailed proof of this result is presented in [7, Corollary 1].

Theorem 2.3. *If a characteristic polynomial of a linear recurrence of order d has m distinct roots $z_1, \dots, z_m$ of multiplicities $d_1, \dots, d_m$ $(d_1 + \cdots + d_m = d)$ then the d sequences*

$$f_{ij}(n) = n^{j-1}z_i^n, \quad 1 \le i \le m, \quad 1 \le j \le d_i, \tag{2.6}$$

form a basis of the vector space V containing the solutions of the recurrence (2.1).

A proof of this results is presented in [7, Theorem 2].

The formulas for the bases of the solution space V expressed in terms of roots of (2.3) help us establish elegant necessary and sufficient conditions for the periodicity of generalized complex Horadam sequences, formulated in terms of matrices. Coupling Lemma 1 to the established bases for the solution space, it is expected that periodic recurrences are closely related to the roots of (2.3) being kth roots of unity, for some $k \in \mathbb{N}$.

3. Main results

When no confusion is possible, the sequence $\{w_n(\mathbf{a};\mathbf{c})\}_{n=0}^{\infty}$ defined in (1.1) for the complex m-tuples $a_1, \dots, a_m$ (initial conditions) and $c_1, \dots, c_m$ (coefficients) is denoted by $\{w_n\}_{n=0}^{\infty}$.

In this section necessary and sufficient conditions for the periodicity of the sequence $\{w_n\}_{n=0}^{\infty}$ are established, when the roots of the characteristic polynomial (2.3) are all distinct, all equal, or when the distinct roots have arbitrary multiplicities. Without loss of generality, it is assumed that the recurrence order cannot be reduced, therefore $c_m \neq 0$.

3.1. m Distinct roots.

Here sufficient and necessary conditions for the periodicity of $\{w_n\}_{n=0}^{\infty}$ are derived, when the roots of the characteristic polynomial (2.3) are all distinct.

Theorem 3.1. *(Sufficient condition for periodicity) Let $z_1, \dots, z_m$ be distinct kth roots of unity $(m \le k)$ and let the polynomial $P(x)$ be defined as*

$$P(x) = (x - z_1)(x - z_2) \cdots (x - z_m), \quad x \in \mathbb{C}. \tag{3.1}$$

The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (3.1) and arbitrary initial conditions

$$w_{i-1} = a_i \in \mathbb{C}, \quad i = 1, \dots, m, \tag{3.2}$$

is periodic.

Proof. The sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (3.1) satisfies the linear recurrence

$$w_n = c_1 w_{n-1} + c_2 w_{n-2} + \cdots + c_m w_{n-m}, \quad m \le n \in \mathbb{N}, \tag{3.3}$$

with the coefficients $c_1, \ldots, c_m$ given by $c_i = (-1)^{i-1} S_i(z_1, \ldots, z_m)$, where $S_i(z_1, \ldots, z_m)$ represents the symmetric sum of products having i (unordered) factors chosen from $z_1, \ldots, z_m$.

From Theorem 2.1, the sequences

$$f_1(n) = z_1^n, f_2(n) = z_2^n, \ldots, f_m(n) = z_m^n,$$

form a basis in the vector space V of solutions of the recurrence (3.3), therefore the n-th term of the sequence can be written as the linear combination

$$w_n = A_1 z_1^n + A_2 z_2^n + \cdots + A_m z_m^n, \tag{3.4}$$

The coefficients $A_1, \ldots, A_m$ can be obtained from (3.4) and the initial conditions (3.2) by solving the system of linear equations

$$\begin{cases} a_1 & = A_1 + A_2 + \cdots + A_m \\ a_2 & = A_1 z_1 + A_2 z_2 + \cdots + A z_m \\ \cdots & \\ a_m & = A_1 z_1^{m-1} + A_2 z_2^{m-1} + \cdots + A z_m^{m-1}. \end{cases}$$

As $z_1, \ldots, z_m$ are kth roots of unity, $z_i^n = z_i^{n+k}$, $i = 1, \ldots, m$, therefore $w_n = w_{n+k}$, $n \in \mathbb{N}$. This shows that the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic and its period divides k. The period is given by the formula $[\text{ord}(z_1), \text{ord}(z_2), \ldots, \text{ord}(z_m)]$, where $\text{ord}(z_i)$ denotes the order of the root z_i and $[b_1, \ldots, b_m]$ denotes the least common multiple (lcm) for a given set of integers $b_1, \ldots, b_m$.

A procedure that allows the direct computation of sequence terms using elementary matrices and operations is detailed below. The initial condition (3.2) can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_m \\ z_1^2 & z_2^2 & \cdots & z_m^2 \\ \vdots & \vdots & & \vdots \\ z_1^{m-1} & z_2^{m-1} & \cdots & z_m^{m-1} \end{pmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = V_{m,m}(\mathbf{z}) \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}, \tag{3.5}$$

where $V_{n,m}(\mathbf{z})$ represents the Vandermonde matrix defined below for $\mathbf{z} = (z_1, \ldots, z_m)$ as

$$V_{n,m}(\mathbf{z}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_m \\ \vdots & \vdots & & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_m^{n-1} \end{pmatrix}. \tag{3.6}$$

As $z_1, \ldots, z_m$ are distinct, the determinant of the square Vandermonde matrix yields [7]

$$\det\Big(V_{m,m}(\mathbf{z})\Big) = \prod_{1 \le i < j \le m} (z_j - z_i) \ne 0.$$

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Using Cramer's rule [9] and the notation $\mathbf{a} = (a_1, \ldots, a_m)$, the unique solution of (3.5) is

$$A_i = \frac{\det\left(V^i_{m,m}(\mathbf{z}, \mathbf{a})\right)}{\det\left(V_{m,m}(\mathbf{z})\right)}, \quad i = 1, \ldots, m, \tag{3.7}$$

where $V^i_{m,m}$ is the matrix obtained by replacing the i-th column of $V_{m,m}(\mathbf{z})$ with $\mathbf{a}^T$.

The first N terms of the sequence $\{w_n\}_{n=0}^{\infty}$ can be computed from the formula

$$V_{N,m}(\mathbf{z}) \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}. \tag{3.8}$$

Fig. 1 illustrates the periodic orbits of sequence $\{w_n\}_{n=0}^{\infty}$ obtained from the recurrence (3.3) (diamonds), or direct formula (3.8) (circles) when selecting (a) $m = 3$ and (b) $m = 5$ distinct roots respectively, from the 7th roots of unity. In the Appendix we present a numerical method allowing the direct computation of sequence terms with a given index set $I = \{i_1, \ldots, i_N\}$. $\square$

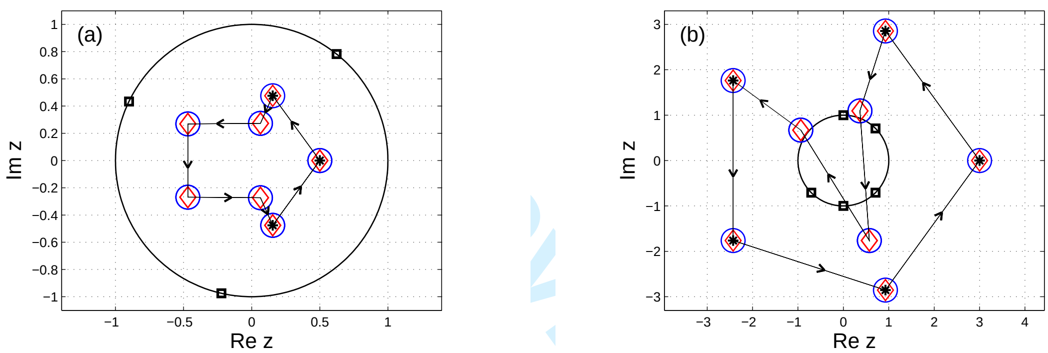

FIGURE 1. First 15 terms of $\{w_n\}_{n=0}^{\infty}$ computed from the recurrence (3.3) (diamonds) and direct formula (3.8) (circles), for (a) $m = 3$, $z_j = e^{\frac{2\pi i}{7}(2j-1)}$, and $a_j = .5e^{\frac{2\pi i}{5}(j+3)}$, $j = 1, 2, 3$; (b) $m = 5$, $z_j = e^{\frac{2\pi i}{7}j}$ for $j = 1, 2, 5, 6, 7$ and $a_j = 3e^{\frac{2\pi i}{5}(j+1)}$, $j = 1, \ldots, 5$. Also plotted, initial conditions (stars), generators (squares) and unit circle S (solid line). Arrows indicate the increase of n.

Theorem 3.2. *(Necessary condition for periodicity) Let us assume that the roots $z_1, \ldots, z_m$ of (3.1) are all distinct. The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (3.1) and initial conditions $a_1, \ldots, a_m$ is periodic only if there exist $k \in \mathbb{N}$ positive s. t.*

$$A_i(z_i^k - 1) = 0, \quad i = 1, \ldots, m, \tag{3.9}$$

where $A_1, \ldots, A_m$ are computed from formula (3.7).

Proof. Let us assume that the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic and let $k \in \mathbb{N}$ be the period. Under this assumption, the periodicity can be written as

$$w_n = w_{n+k}, \quad \forall n \in \mathbb{N}. \tag{3.10}$$

For distinct roots $z_1, \dots, z_m$ the formula (3.4) of the general term w_n and (3.10) give

$$w_n = A_1 z_1^n + A_2 z_2^n + \cdots + A_m z_m^n = A_1 z_1^{n+k} + A_2 z_2^{n+k} + \cdots + A_m z_m^{n+k} = w_{n+k}, \quad \forall n \in \mathbb{N},$$

or equivalently

$$w_{n+k} - w_n = A_1(z_1^k - 1)z_1^n + A_2(z_2^k - 1)z_2^n + \cdots + A_m(z_m^k - 1)z_m^n = 0, \forall n \in \mathbb{N}. \tag{3.11}$$

Writing (3.11) for $n = 0, \dots, m-1$ one obtains the system

$$V_{m,m}(\mathbf{z}) \begin{bmatrix} A_1(z_1^k - 1) \\ A_2(z_2^k - 1) \\ \vdots \\ A_m(z_m^k - 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.12}$$

As $\det V_{m,m}(\mathbf{z}) \neq 0$, the system has the unique solution

$$A_i(z_i^k - 1) = 0, \; i = 1, \dots, m.$$

For each $i = 1, \dots, m$ there are two possibilities: either z_i is a kth root of unity, or it doesn't appear in the formula of the general term (3.4) (in this case A_i is zero). The only non-degenerate solution of (3.11) (when each generator has a non-zero contribution) satisfies

$$z_1^k = z_2^k = \cdots = z_m^k = 1,$$

in which case $z_1, \dots, z_m$ are distinct kth roots of unity. □

3.2. **One root of multiplicity m.** Here sufficient and necessary conditions for the periodicity of $\{w_n\}_{n=0}^\infty$ are presented when the m roots of the characteristic polynomial (2.3) are all equal.

Theorem 3.3. *(Sufficient condition for periodicity) Let z be a kth root of unity, m a natural number and let the polynomial $P(x)$ be defined as*

$$P(x) = (x - z)^m, \quad x \in \mathbb{C}. \tag{3.13}$$

The recurrent sequence $\{w_n\}_{n=0}^\infty$ generated by the characteristic polynomial (3.13) and initial conditions (3.2) is periodic when

$$A_2 = A_3 = \cdots = A_m = 0,$$

where $A_1, \dots, A_m$ are the coefficients of $\{w_n\}_{n=0}^\infty$ in the basis (2.5) defined in Theorem 2.2. The sequence $\{w_n\}_{n=0}^\infty$ is divergent otherwise.

Proof. Similarly to Theorem 3.1, the sequence $\{w_n\}_{n=0}^\infty$ satisfies the linear recurrence (3.3), for the coefficients $c_1, \dots, c_m$ given by

$$c_i = (-1)^{i-1} \binom{m}{i} z^i, \quad i = 1, \dots, m. \tag{3.14}$$

From Theorem 2.2, the sequences

$$f_1(n) = z^n, f_2(n) = n z_2^n, \dots, f_m(n) = n^{m-1} z^n,$$

form a basis in the vector space V of solutions of the recurrence generated by the characteristic polynomial (3.13), therefore the n-th term of the sequence can be written as

$$w_n = A_1 z^n + n A_2 z^n + \cdots + n^{m-1} A_m z^n. \tag{3.15}$$

For $A_i = 0$, $i = 2, \dots, m$ we have $w_n = A_1 z^n$, so in this case the sequence $\{w_n\}_{n=0}^\infty$ is periodic. Whenever there exists $i \geq 2$ s.t. $A_i \neq 0$, the behaviour of w_n is dictated by the divergent coefficient $n^i A_i$ of z^n, therefore the sequence $\{w_n\}_{n=0}^\infty$ diverges.

This property is illustrated in Fig. 2, where it is shown that the sequence can either be (a) periodic or (b) divergent. Some asymptotic properties of divergent sequences generated by roots of unity are presented in §4.1.

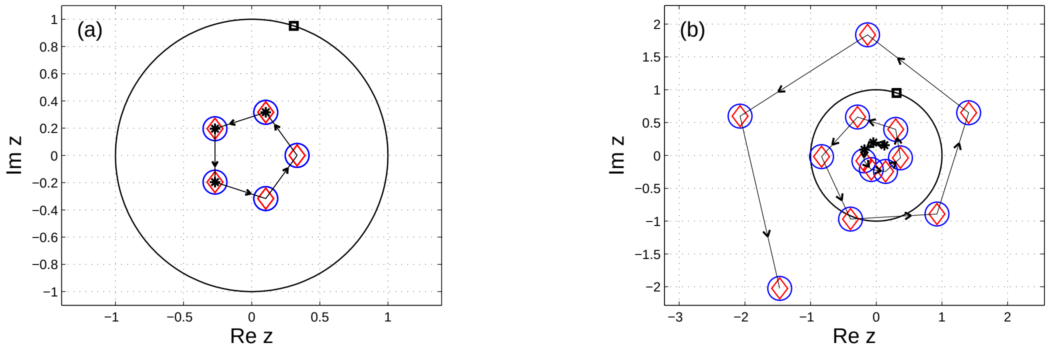

FIGURE 2. First 16 terms of sequence $\{w_n\}_{n=0}^{\infty}$ computed from the recurrence (3.3) (diamonds) and direct formula (3.19) (circles) for $m = 3$, $z = e^{\frac{2\pi i}{5}}$ and initial conditions (a) $a_j = e^{\frac{2\pi i}{5}j}/3$, $j = 1, 2, 3$; (b) $a_j = e^{\frac{2\pi i}{7}j}/3$, $j = 1, 2, 3$. Also plotted are initial conditions (stars), generators (squares) and unit circle S (solid line). Arrows indicate the increase of the sequence index n.

The coefficients $A_1, \dots, A_m$ are obtained by using (3.15) to write the initial conditions as

$$\begin{cases} a_1 & = A_1 \\ a_2 & = A_1 + A_2 + \cdots + A_m \\ \dots & \\ a_m & = A_1 z^{m-1} + (m-1)A_2 z^{m-1} + \cdots + (m-1)^{m-1} A z^{m-1}. \end{cases}$$

A procedure that allows the direct computation of sequence terms using elementary matrices and operations is detailed below. The initial condition can be written in matrix form as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ z & z & \cdots & z \\ z^2 & 2z^2 & \cdots & 2^{m-1}z^2 \\ \vdots & \vdots & & \vdots \\ z^{m-1} & (m-1)z^{m-1} & \cdots & (m-1)^{m-1}z^{m-1} \end{pmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \mathcal{V}_{m,m}(z) \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}, \quad (3.16)$$

where the matrix $\mathcal{V}_{n,m}(z)$ is defined for the integers n, m and the complex number z by

$$\mathcal{V}_{n,m}(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ z & z & \cdots & z \\ z^2 & 2z^2 & \cdots & 2^{m-1}z^2 \\ \vdots & \vdots & & \vdots \\ z^{n-1} & (n-1)z^{n-1} & \cdots & (n-1)^{m-1}z^{n-1} \end{pmatrix}. \quad (3.17)$$

By collecting the powers of z in each row, the determinant of (3.17) for $n = m$ is

$$\det \mathcal{V}_{n,n}(z) = z^{\frac{n(n-1)}{2}} \det\left(V_{n,n}^T(0,\ldots,n-1)\right) = z^{\frac{n(n-1)}{2}} \prod_{0\le i<j\le n-1} (j-i) \neq 0,$$

where $V_{n,n}^T$ denotes the transpose of the Vandermonde matrix defined in (3.6).

Simple computations show that (3.17) can be computed numerically as

$$\mathcal{V}_{n,m}(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & n-1 & \cdots & (n-1)^{m-1} \end{pmatrix} .* \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z & z & \cdots & z \\ z^2 & z^2 & \cdots & z^2 \\ \vdots & \vdots & & \vdots \\ z^{n-1} & z^{n-1} & \cdots & z^{n-1} \end{pmatrix}.$$

where $.*$ denotes the element-by-element product of matrices from Matlab®, and expressed in terms of Vandermonde matrices as follows

$$\mathcal{V}_{n,m}(z) = V_{n,m}(z,\ldots,z). * V_{m,n}^T(0,1,\ldots,n-1).$$

From Cramer's rule and $\det \mathcal{V}_{m,m} \neq 0$, the system (3.16) has the unique solution

$$A_i = \frac{\det\left(\mathcal{V}_{m,m}^i(z,\mathbf{a})\right)}{\det \mathcal{V}_{m,m}(z)}, \quad i = 1,\ldots,m, \tag{3.18}$$

where $\mathcal{V}_{m,m}^i$ is the matrix obtained by replacing the i-th column of $\mathcal{V}_{m,m}(z)$ with $\mathbf{a}^T$.

The first N terms of the sequence $\{w_n\}_{n=0}^{\infty}$ can be obtained using the formula

$$\mathcal{V}_{N,m}(z) \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}. \tag{3.19}$$

The terms of the sequence $\{w_n\}_{n=0}^{\infty}$ can therefore be obtained in two alternative ways: either from the recurrence (3.3) using the coefficients given by (3.14), or directly from formula (3.19). In the Appendix we present a numerical method that allows the direct computation of sequence terms with a given index set $I = \{i_1,\ldots,i_N\}$. □

Theorem 3.4. *(Necessary condition for periodicity) The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (3.13) and initial conditions (3.2) is only periodic when*

$$A_1(z^k - 1) = 0, A_2 = A_3 = \cdots = A_m = 0, \tag{3.20}$$

where $A_1,\ldots,A_m$ are computed from formula (3.18).

Proof. Let us assume that the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic and let $k \in \mathbb{N}$ be the period. The periodicity condition and formula (3.15) for the general term of $\{w_n\}_{n=0}^{\infty}$ give

$$\begin{aligned} w_n &= (A_1 + nA_2 + \cdots + n^{m-1}A_m)z^n \\ &= (A_1 + (n+k)A_2 + \cdots + (n+k)^{m-1}A_m)z^{n+k} = w_{n+k}, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.21}$$

Writing the periodicity condition as $w_{n+k} - w_n = 0$, one obtains

$$\left[A_1\left(z^k - 1\right) + A_2\left(z^k(n+k) - n\right) + \cdots + A_m\left(z^k(n+k)^{m-1} - n^{m-1}\right)\right]z^n = 0, \quad \forall n \in \mathbb{N}.$$

For a fixed value of $z \in \mathbb{C}$, this equation represents a polynomial of degree $m-1$ in n

$$Q(n) = B_0 + B_1 n + \cdots + B_{m-1} n^{m-1} = 0, \quad \forall n \in \mathbb{N}.$$

As $Q(n)$ has infinitely many zeros, this has to be a null polynomial, therefore

$$\begin{cases} B_0 & = A_1\left(z^k - 1\right) + A_2 k z^k + A_3 k^2 z^k + \cdots + A_m k^{m-1} z^k = 0, \\ B_1 & = A_2\left(z^k - 1\right) + A_3 \binom{2}{1} k z^k + A_4 \binom{3}{1} k^2 z^k + \cdots + A_m \binom{m-1}{1} k^{m-2} z^k = 0, \\ \cdots & \\ B_j & = A_{j+1}\left(z^k - 1\right) + A_{j+2}\binom{j+1}{j} k z^k + \cdots + A_m \binom{m-1}{j} k^{m-1-j} z^k = 0, \\ \cdots & \\ B_{m-1} & = A_m\left(z^k - 1\right) = 0. \end{cases}$$

The above system of linear equations in $A_1, A_2, \ldots, A_m$ can be written in matrix form as

$$\begin{pmatrix} z^k - 1 & k z^k & k^2 z^k & \cdots & k^{m-1} z^k \\ 0 & z^k - 1 & 2k z^k & \cdots & (m-1) k^{m-2} z^k \\ 0 & 0 & z^k - 1 & \cdots & \binom{m-1}{2} k^{m-3} z^k \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z^k - 1 \end{pmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.22}$$

and two distinct situations emerge.

1. If $z^k - 1 \neq 0$ the determinant of the homogeneous system (3.22) is not zero, therefore the system has the unique solution $A_1 = \cdots = A_m = 0$.
2. When $z^k = 1$ the above system is reduced to

$$\begin{pmatrix} k & k^2 & k^3 & \cdots & k^{m-1} \\ 0 & 2k & 3k^2 & \cdots & (m-1) k^{m-2} \\ 0 & 0 & 3k & \cdots & \binom{m-1}{2} k^{m-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \binom{m-1}{m-2} k \end{pmatrix} \begin{bmatrix} A_2 \\ A_3 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

As the determinant of the matrix is $(m-1)! k^m \neq 0$, the above homogeneous system has the unique solution $A_2 = \cdots = A_m = 0$, while A_1 can be arbitrary.

Both situations are captured in formula (3.20). □

3.3. **m distinct roots of multiplicities $d_1, \ldots, d_m$.** Here sufficient and necessary conditions for the periodicity of recurrence $\{w_n\}_{n=0}^{\infty}$ are presented, for sequences having characteristic polynomials with roots of different multiplicities.

Theorem 3.5. *(Sufficient condition for periodicity) Let $2 \leq m \leq k$ and $d_1, \ldots, d_m$ be natural numbers, $z_1, \ldots, z_m$ distinct kth roots of unity and let the polynomial $P(x)$ be defined as*

$$P(x) = (x - z_1)^{d_1} (x - z_2)^{d_2} \cdots (x - z_m)^{d_m}, \quad x \in \mathbb{C}. \tag{3.23}$$

The linear recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (3.23) of degree $d = d_1 + \cdots + d_m$ and initial conditions $w_{i-1} = a_i$, $i = 1, \ldots, d$ is periodic when

$$A_{ij} = 0, \quad 1 \leq i \leq m, \quad 2 \leq j \leq d_i, \tag{3.24}$$

where A_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq d_i)$ represent the coefficients of $\{w_n\}_{n=0}^{\infty}$ in the basis (2.6) defined in Theorem 2.3. The sequence $\{w_n\}_{n=0}^{\infty}$ is divergent otherwise.

Proof. The sequence $\{w_n\}_{n=0}^{\infty}$ satisfies the linear recurrence of order d below

$$w_n = c_1 w_{n-1} + c_2 w_{n-2} + \cdots + c_d w_{n-d}, \quad d \le n \in \mathbb{N}, \tag{3.25}$$

whose coefficients are given by $c_i = (-1)^{i-1} S_i(Z)$, $i = 1, \ldots, d$ where $S_i(Z)$ represents the symmetric sum of products having i factors chosen from the set

$$Z = \{\underbrace{z_1, \ldots, z_1}_{d_1}, \underbrace{z_2, \ldots, z_2}_{d_2}, \ldots, \underbrace{z_m, \ldots, z_m}_{d_m}\}.$$

From Theorem 2.3, the sequences

$$f_{ij}(n) = n^{j-1} z_i^n, \quad 1 \le i \le m, \quad 1 \le j \le d_i, \tag{3.26}$$

form a basis in the vector space V of solutions of the recurrence (3.25), therefore the nth term of the sequence can be written as

$$w_n = \sum_{i=1}^{m} \left(A_{i1} + nA_{i2} + \cdots + n^{d_i - 1} A_{id_i} \right) z_i^n. \tag{3.27}$$

When the condition (3.24) is fulfilled, the representation (3.27) of w_n reduces to

$$w_n = A_{11} z_1^n + A_{21} z_2^n + \cdots + A_{m1} z_m^n,$$

which is similar to (3.4) discussed in Theorem 3.1, which is periodic for distinct $z_1, \ldots, z_m$. As illustrated in Fig. 3 (a), the sequence may be periodic even when $d_i \ge 2$, $i = 1, \ldots, m$. When any of the coefficients A_{ij}, $1 \le i \le m$, $2 \le j \le d_i$ does not vanish, the polynomial (3.27) in n is not constant, therefore the sequence $\{w_n\}_{n=0}^{\infty}$ diverges as depicted in Fig. 3 (b). A detailed explanation of this behaviour is presented in Theorem 3.6.

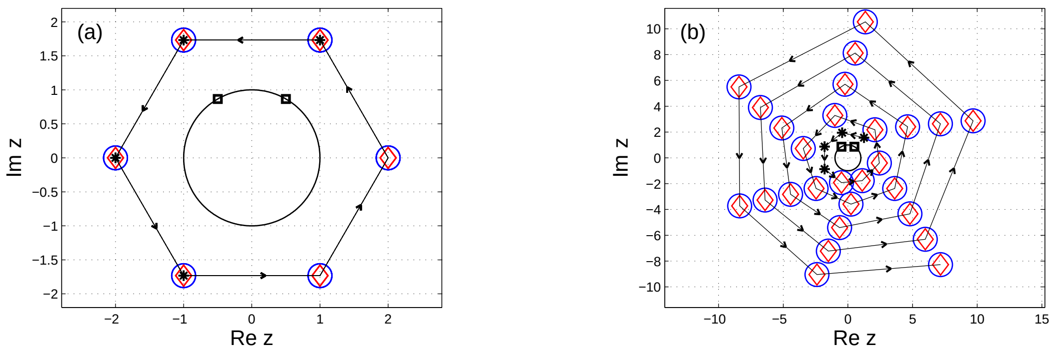

FIGURE 3. First 31 terms of sequence $\{w_n\}_{n=0}^{\infty}$ computed from the recurrence (3.25) (diamonds) and direct formula (3.31) (circles) for $z_1 = e^{\frac{2\pi i}{6}}$, $z_2 = e^{\frac{4\pi i}{6}}$, $d_1 = d_2 = 2$ and initial conditions (a) $a_j = 2e^{\frac{2\pi i}{6} j}$, $j = 1, \ldots, 4$; (b) $a_j = 2e^{\frac{2\pi i}{7} j}$, $j = 1, \ldots, 4$. Also plotted are initial conditions (stars), generators (squares) and unit circle S (solid line). Arrows indicate the increase of sequence index.

A procedure allowing the computation of sequence terms using elementary matrices is detailed below. For $m, n \in \mathbb{N}$, $\mathbf{z} = (z_1, \ldots, z_m)$ and $\mathbf{d} = (d_1, \ldots, d_m)$ one can define the matrix

$$\mathcal{W}_{n+1,\,d}(\mathbf{z},\mathbf{d}) = \left(\begin{array}{cccc|c|cccc} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ z_1 & z_1 & \cdots & z_1 & \cdots & z_m & z_m & \cdots & z_m \\ z_1^2 & 2z_1^2 & \cdots & 2^{d_1-1}z_1^2 & \cdots & z_m^2 & 2z_m^2 & \cdots & 2^{d_m-1}z_m^2 \\ \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\ z_1^n & nz_1^n & \cdots & n^{d_1-1}z_1^n & \cdots & z_m^n & nz_m^n & \cdots & n^{d_m-1}z_m^n \end{array}\right). \tag{3.28}$$

The above matrix can be constructed using the matrices $\mathcal{V}_{n,d_i}(z_i), i = 1, \ldots, m$ (3.17), as

$$\mathcal{W}_{n,\,d}(\mathbf{z},\mathbf{d}) = \left(\; \mathcal{V}_{n,d_1}(z_1) \;\middle|\; \cdots \;\middle|\; \mathcal{V}_{n,d_m}(z_m) \;\right).$$

The columns of the above matrix represent the first $n+1$ terms of each member of basis (3.26) so they are linearly independent, therefore for $d = n+1$ we have $\det \mathcal{W}_{d,\,d} \neq 0$.

The coefficients A_{ij}, $1 \leq i \leq m$, $1 \leq j \leq d_i$ are computed from the initial condition $w_{i-1} = a_i$, $i = 1, \ldots, d$, written with the help of formula (3.27) as the linear system

$$\mathcal{W}_{d,\,d}(\mathbf{z},\mathbf{d}) \left(\; A_{11} \quad \cdots \quad A_{1d_1} \;\middle|\; \cdots \;\middle|\; A_{m1} \quad \cdots \quad A_{md_m} \;\right)^T = (a_1, a_2, \ldots, a_d)^T. \tag{3.29}$$

Using Cramer's rule and that $\det \mathcal{W}_{d,\,d} \neq 0$, the above system has the unique solution

$$A_{ij} = \frac{\det\left(\mathcal{W}^l_{d,\,d}(\mathbf{z},\mathbf{d},\mathbf{a})\right)}{\det\left(\mathcal{W}_{d,\,d}(\mathbf{z},\mathbf{d})\right)}, \quad i = 1, \ldots, m, \tag{3.30}$$

where $\mathcal{W}^l_{d,\,d}(\mathbf{z},\mathbf{d},\mathbf{a})$ is the matrix obtained by replacing the lth column of $\mathcal{W}_{d,\,d}(\mathbf{z},\mathbf{d})$ by $\mathbf{a}^T$, with l computed from the formula $l = d_1 + \cdots + d_{i-1} + j$.

The first N terms of the sequence $\{w_n\}_{n=0}^{\infty}$ can therefore be obtained from the formula

$$\mathcal{W}_{N,\,d}(\mathbf{z},\mathbf{d}) \left(\; A_{11} \quad \cdots \quad A_{1d_1} \;\middle|\; \cdots \;\middle|\; A_{m1} \quad \cdots \quad A_{md_m} \;\right)^{\mathbf{T}} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}. \tag{3.31}$$

The terms of the sequence $\{w_n\}_{n=0}^{\infty}$ can be obtained in two alternative ways: from the recurrence (3.25), or directly from (3.31). In the Appendix we present a numerical method allowing the direct computation of sequence terms with a given index set $I = \{i_1, \ldots, i_N\}$. □

Theorem 3.6. *(Necessary condition for periodicity) Let $2 \leq m \leq k$ and $d_1, \ldots, d_m$ be natural numbers. The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (3.23) of degree $d = d_1 + \cdots + d_m$ and initial conditions $w_{i-1} = a_i$, $i = 1, \ldots, d$ is periodic only if*

$$\begin{aligned} A_{i1}(z_i^k - 1) &= 0, \quad i = 1, \ldots, m, \\ A_{ij} &= 0, \quad j = 2, \ldots, d_i, \end{aligned} \tag{3.32}$$

where the coefficients A_{ij} are computed from formula (3.30), for $1 \leq i \leq m$, $1 \leq j \leq d_i$.

Proof. Assuming the sequence has periodicity $k \in \mathbb{N}$, the formula (3.27) of the general term for $\{w_n\}_{n=0}^{\infty}$ and the periodicity condition $w_{n+k} - w_n = 0$ give

$$\sum_{i=1}^{m} \left[A_{i1} \left(z_i^k - 1 \right) + A_{i2} \left(z_i^k (n+k) - n \right) + \cdots + A_{id_i} \left(z_i^k (n+k)^{d_i - 1} - n^{d_i - 1} \right) \right] z_i^n = 0, \tag{3.33}$$

which holds for $n \in \mathbb{N}$. From Theorem 2.3, the sequences $\{z_1^n\}_{n=0}^{\infty}, \ldots, \{z_m^n\}_{n=0}^{\infty}$ are linearly independent, therefore their coefficients in (3.33) are zero, which gives

$$A_{i1} \left(z_i^k - 1 \right) + A_{i2} \left(z_i^k (n+k) - n \right) + \cdots + A_{id_i} \left(z_i^k (n+k)^{d_i - 1} - n^{d_i - 1} \right) = 0, \tag{3.34}$$

for $i = 1, \ldots, m$ and $n \in \mathbb{N}$. As seen in the case of a single root with arbitrary multiplicity presented in Theorem 3.4, for each value $i = 1, \ldots, m$ the only solutions of (3.34) are

1. $z_i^k \neq 1$ when $A_{i1} = A_{i2} = \cdots = A_{id_i} = 0$,
2. $z_i^k = 1$ when A_{i1} is arbitrary and $A_{i2} = \cdots = A_{id_i} = 0$.

The only non-degenerate solution of (3.33) (each root appears explicitly) satisfies

$$z_1^k = z_2^k = \cdots = z_m^k = 1,$$

in which case $z_1, \ldots, z_m$ are distinct kth roots of unity. □

4. Further results

In this section we examine the asymptotic behaviour of sequences generated by roots of unity and examine inner and boundaries for the regions containing periodic orbits.

4.1. **Asymptotic behaviour of subsequences.** This discussion extends the result regarding periodic sequences presented in Theorem 3.5, with the analysis of the asymptotic behaviour of divergent sequences generated by repeated roots of unity.

Theorem 4.1. *Let $2 \leq m \leq k$ and $d_1, \ldots, d_m$ be natural numbers, $z_1, \ldots, z_m$ distinct kth roots of unity and the sequence $\{w_n\}_{n=0}^{\infty}$ be generated by the characteristic polynomial (3.23). Defining the number*

$$d^* = \max\{j : A_{ij} \neq 0, i \in \{1, \ldots, m\}\}, \tag{4.1}$$

for the coefficients A_{ij} ($1 \leq i \leq m$, $1 \leq j \leq d_i$) given in (3.27), one obtains the following properties referring to the sequence $\{w_n\}_{n=0}^{\infty}$ and the subsequences $\{w_{Nk+j}\}_{N=0}^{\infty}$:

(a) *For $d^* \leq 1$ the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic.*
(b) *For $d^* \geq 2$ the sequence $\{w_n\}_{n=0}^{\infty}$ is divergent.*
(c) *For $d^* \leq 2$ the terms of $\{w_{Nk+j}\}_{N=0}^{\infty}$ are collinear (including the periodic case $d^* = 1$).*
(d) *For $d^* \geq 3$ the terms of $\{w_{Nk+j}\}_{N=0}^{\infty}$ converge asymptotically towards straight lines.*

Proof.
(a) This is the subject of Theorem 3.1.
(b) Denote by $I_{d^*} \subset \{1, \ldots, m\}$ the index set for which the maximum d^* is attained in (4.1). From Theorem 2.3, the sequences $\{z_1^n\}_{n=0}^{\infty}, \ldots, \{z_m^n\}_{n=0}^{\infty}$ are linearly independent, so implicitly $\{z_i^n\}_{n=0}^{\infty}$, $i \in I_{d^*}$ are linearly independent. The term w_n given by formula (3.27) contains a monomial of degree $d^* - 1$ in n, written as

$$\sum_{i \in I_{d^*}} A_{id^*} n^{d^*-1} z_i^n = n^{d^*-1} \sum_{i \in I_{d^*}} A_{id^*} z_i^n.$$

We show that the coefficients of n^{d^*-1} in w_n cannot be all zero if $d^* \geq 2$.

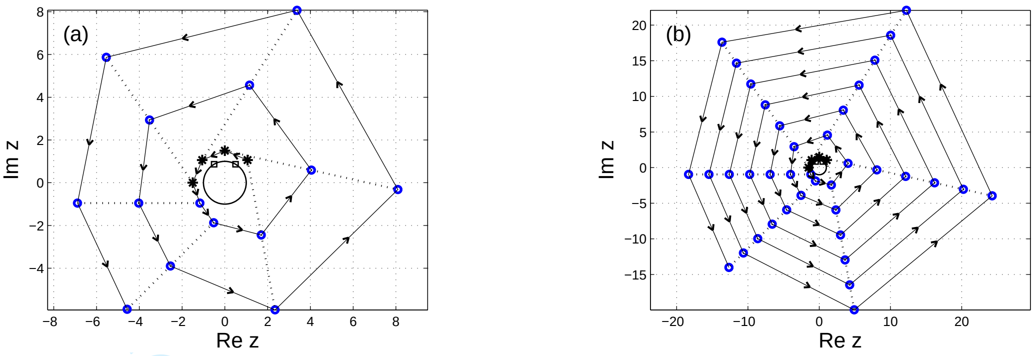

FIGURE 4. First (a) $N = 18$; (b) $N = 42$ terms of $\{w_n\}_{n=0}^{\infty}$ computed from the direct formula (3.31) (circles) for generators $z_j = e^{\frac{2\pi i}{6}j}$, $j = 1, 2$, multiplicities $d^* = d_1 = d_2 = 2$ and initial conditions $a_j = 2e^{\frac{2\pi i}{8}j}$, $j = 1, \ldots, 4$. Also plotted, initial conditions (stars), generators (squares) and unit circle S (solid line). Arrows indicate the increase of n.

As $z_1, \ldots, z_m$ are kth roots of unity, the following relation holds

$$\sum_{i \in I_{d^*}} A_{id^*} z_i^n = \sum_{i \in I_{d^*}} A_{id^*} z_i^{n+k},$$

therefore the above expression can only have one of the values

$$\sum_{i \in I_{d^*}} A_{id^*}, \sum_{i \in I_{d^*}} A_{id^*} z_i, \ldots, \sum_{i \in I_{d^*}} A_{id^*} z_i^{k-1}.$$

If these values are all zero one obtains $\sum_{i \in I_{d^*}} A_{id^*} z_i^n = 0$, and the linear independence of vectors $\{z_i^n\}_{n=0}^{\infty}$ $(i \in I_{d^*})$ gives $A_{id^*} = 0$ $(i \in I_{d^*})$, which contradicts the definition (4.1) of d^*. There is therefore an index $j \in \{0, \ldots, k-1\}$ for which $\sum_{i \in I_{d^*}} A_{id^*} z_i^j$ is not zero, so

$$n^{d^*-1} \sum_{i \in I_{d^*}} A_{id^*} z_i^{Nk+j} = n^{d^*-1} \sum_{i \in I_{d^*}} A_{id^*} z_i^j$$

is divergent. As this is the leading term of w_{Nk+j} in powers of n, $\{w_n\}_{n=0}^{\infty}$ is also divergent.

(c) For each $j \in \{0, \ldots, k-1\}$ and $N \in \mathbb{N}$ we have $z_i^{Nk+j} = z_i^j$, therefore (3.27) gives

$$\begin{aligned} w_{Nk+j} = &\left(A_{11} + (Nk+j)A_{12} + \cdots + (Nk+j)^{d_1-1} A_{1d_1}\right) z_1^j + \cdots + \\ &\left(A_{m1} + (Nk+j)A_{m2} + \cdots + (Nk+j)^{d_m-1} A_{md_m}\right) z_m^j. \end{aligned} \tag{4.2}$$

For a fixed value of j, (4.2) represents a polynomial of degree $d^* - 1$ in N, written as

$$w_{Nk+j} = B_1 + B_2(Nk+j) + \cdots + (Nk+j)^{d^*-1} B_{d^*}, \tag{4.3}$$

where d^* was defined in (4.1). For $d^* = 2$ we obtain

$$w_{Nk+j} - w_j = [B_1 + B_2(Nk+j)] - [B_1 + B_2 j\,] = NkB_2, \tag{4.4}$$

whose argument is independent of N. The terms of the subsequence $\{w_{Nk+j}\}_{N=0}^{\infty}$ are then collinear for each value of $j \in \{0, \ldots, k-1\}$. For $d^* = 1$ the condition $w_{Nk+j} - w_j = 0$ confirms the periodicity. The behaviour of initial terms of subsequences is illustrated in Fig. 4 (a).

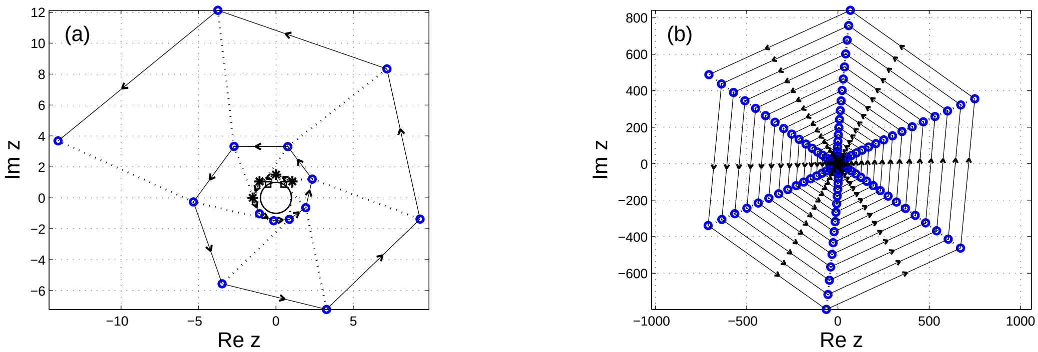

FIGURE 5. First (a) $N = 18$; (b) $N = 120$ terms of sequence $\{w_n\}_{n=0}^{\infty}$ computed from the direct formula (3.31) (circles) for generators $z_j = e^{\frac{2\pi i}{6}j}$, $j = 1, 2$, multiplicities $d^* = d_1 = 3$, $d_2 = 1$ and initial conditions $a_j = 2e^{\frac{2\pi i}{8}j}$, $j = 1, \ldots, 4$. Also plotted are initial conditions (stars), generators (squares) and unit circle S (solid line). Arrows indicate the increase of n.

(d) For finding the geometrical properties of the sequences $\{w_{Nk+j}\}_{N=0}^{\infty}$ when $d^* \geq 3$ one needs to examine the argument of the general term (4.3). This verifies

$$\begin{aligned}\arg(w_{Nk+j}) &= \arg\left(B_1 + B_2(Nk+j) + \cdots + (Nk+j)^{d^*-1}B_{d^*}\right)\\ &= \arg\left[\frac{1}{(Nk+j)^{d^*-1}}\left(B_1 + B_2(Nk+j) + \cdots + (Nk+j)^{d^*-1}B_{d^*}\right)\right]. \qquad (4.5)\end{aligned}$$

In the limit $N \to \infty$, the argument of w_{Nk+j} satisfies

$$\begin{aligned}\lim_{N\to\infty} \arg(w_{Nk+j}) &= \lim_{N\to\infty} \arg\left[\frac{1}{(Nk+j)^{d^*-1}}\left(B_1 + B_2(Nk+j) + \cdots + (Nk+j)^{d^*-1}B_{d^*}\right)\right]\\ &= \arg\left[\lim_{N\to\infty}\frac{1}{(Nk+j)^{d^*-1}}\left(B_1 + B_2(Nk+j) + \cdots + (Nk+j)^{d^*-1}B_{d^*}\right)\right]\\ &= \arg\left(B_{d^*}\right). \qquad (4.6)\end{aligned}$$

This shows that the terms of the subsequence $\{w_{Nk+j}\}_{N=0}^{\infty}$ align asymptotically to the line of argument $\arg(B_{d^*})$, defined for every value of $j \in \{0, \ldots, k-1\}$. The asymptotic behaviour of subsequences is sketched in Fig. 5.

4.2. **Boundaries of periodic orbits.** Here the triangle inequality is used to derive outer boundaries of the regions containing the orbits of periodic Horadam sequences.

Theorem 4.2. *Let $2 \leq m \leq k$ and $d_1, \ldots, d_m$ be natural numbers, $z_1, \ldots, z_m$ the distinct roots of the polynomial $P(x)$ defined in (3.23) for the complex numbers $a_1, \ldots, a_m$ (initial conditions) and $c_1, \ldots, c_m$ (recurrence coefficients) be complex numbers. Let $\{w_n(\boldsymbol{a}; \mathbf{c})\}_{n=0}^{\infty}$ defined in (3.25) be the associated generalized Horadam sequence. When periodic, the orbit of the Horadam sequence is located inside the disk of radius $|A_{11}| + |A_{21}| + \cdots + |A_{m1}|$, where the coefficients A_{j1}, $j = 1, \ldots, m$ are computed from formula (3.29).*

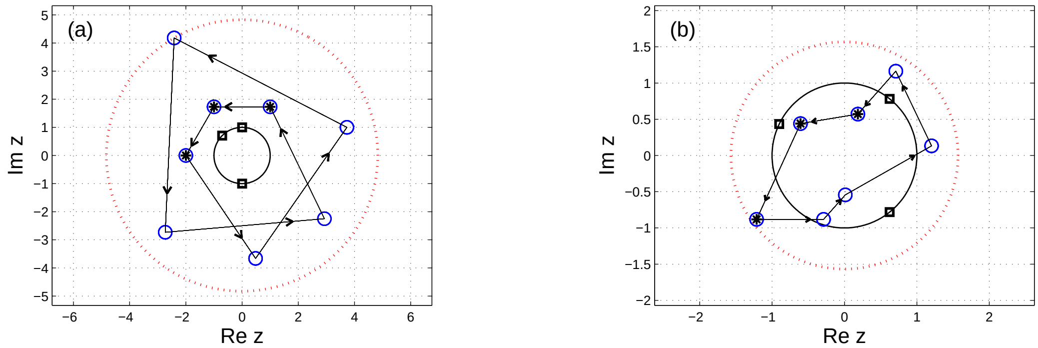

FIGURE 6. Orbit of sequence $\{w_n\}_{n=0}^{\infty}$ computed from formula (3.27) (circles) for $m = 3$ distinct roots of multiplicity one (a) $z_1 = e^{2\pi i\frac{2}{8}}$, $z_2 = e^{2\pi i\frac{3}{8}}$, $z_3 = e^{2\pi i\frac{6}{8}}$ and initial conditions $a_j = 2e^{\frac{2\pi i}{6}j}$, $j = 2, 3, 6$; (b) $z_1 = e^{\frac{2\pi i}{7}}$, $z_2 = e^{\frac{6\pi i}{7}}$, $z_3 = e^{\frac{12\pi i}{7}}$ and initial conditions $a_1 = 3/5e^{\frac{2\pi i}{5}}$, $a_2 = 3/4e^{\frac{4\pi i}{5}}$, $a_3 = 3/2e^{\frac{6\pi i}{5}}$. Also plotted are initial conditions $a_1, \dots, a_m$ (stars), generators $z_1, \dots, z_m$ (squares), unit circle S (solid line) and the circle $S(0,\, |A_{11}| + |A_{21}| + \cdots + |A_{m1}|)$ (dotted line).

Proof. Any two complex numbers u and v verify the triangle inequality [15, p.18]

$$\big|\, |u| - |v| \,\big| \leq |u + v| \leq |u| + |v|, \tag{4.7}$$

and in general, for any complex numbers $x_1, \dots, x_m$ the following inequality holds

$$|x_1 + \cdots + x_m| \leq |x_1| + \cdots + |x_m|. \tag{4.8}$$

As follows from Theorem 3.6, the periodic solutions can be represented as

$$w_n = A_{11}z_1^n + A_{21}z_2^n + \cdots + A_{m1}z_m^n. \tag{4.9}$$

Using (4.8), (4.9) and $|z_1| = |z_2| = \cdots = |z_m| = 1$ one obtains

$$|w_n| \leq |A_{11}z_1^n| + |A_{21}z_2^n| + \cdots + |A_{m1}z_m^n| = |A_{11}| + |A_{21}| + \cdots + |A_{m1}|,$$

which ends the proof. We illustrate this result in Fig. 6. □

Remark 4.3 For $m = 2$ one can use the left-hand side of (4.7) to obtain a lower bound for the orbit, as illustrated in [1, Theorem 4.1].

Remark 4.4 For $m \geq 3$ establishing an inner bound for the periodic orbit is generally not possible. Various inequalities for the left hand-side of (4.7) that only involve $|x_1|, \cdots, |x_m|$ are presented in the monograph of Dragomir [4, Chapter 3], but under restrictive assumptions on $|x_1|, \cdots, |x_m|$. The main reason for this can be identified even for $m = 3$, where the sum $x_1 + x_2 + x_3$ can vanish while the radii $|x_1|$, $|x_2|$ and $|x_3|$ may not be related by an identity.

When $x_1, \dots, x_m$ are real the following result (left as an exercise for the reader) holds.
Remark 4.5 Let $m \geq 2$ and $x_1, \dots, x_m$ be real numbers (or complex but aligned). Then the following inequality holds

$$\min \big|\pm |x_1| \pm |x_2| \pm \cdots \pm |x_m|\big| \leq |x_1 + \cdots + x_m| \leq |x_1| + \cdots + |x_m|. \tag{4.10}$$

5. Conclusions

In this paper the periodicity of generalized Horadam sequences produced by higher-order linear recurrences with complex coefficients and arbitrary initial conditions was characterized. The necessary and sufficient conditions for the periodicity have been formulated in terms of special matrices, which were built using standard Matlab commands (code provided in the Appendix). The asymptotic behaviour of generalized Horadam sequences generated by roots of unity has also been examined and upper boundaries for the periodic sequences have been established. Throughout, generalized orbits were visualized in the complex plane.

This study breaks the ground for a significant number of further results and applications. An immediate step would be an investigation into the geometric structure of periodic orbits produced by generalized Horadam sequences, which would extend the results obtained in [2]. For given initial conditions and recurrence coefficients, one may enumerate all the generalized Horadam sequences having a given period. The basic principles for this analysis have been presented in [3], where Horadam sequences with a given period were enumerated.

Geometric patterns related to the Fibonacci numbers were linked to optimal solutions for the layout of mirrors in a concentrated solar power plant [10]. Generalized periodic Horadam orbits are expected to have applications in the efficient distribution of data (or files, programmes) in networks or the design of random number generators matching geometric patterns [11]. Recurrent sequences of given length may also be used in the study of multi-phase signals [12].

Acknowledgement

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6. Appendix

The methods presented in §3.1−3.3 for generating the first terms of the generalized Horadam sequence can be refined to allow the direct computation of the sequence terms with a given index set $I = \{i_1, \dots, i_N\}$. We detail below the computer algorithm, using matrix operations implemented in Matlab®.

Distinct roots. In this case matrix $V_{N,m}(z_1, \dots, z_m)$ in (3.8) can be replaced by the matrix $V_{I,m}(z_1, \dots, z_m)$ defined for each set $I = \{i_1, \dots, i_N\}$ as

$$V_{I,m}(z_1, \dots, z_m) = \begin{pmatrix} z_1^{i_1} & z_2^{i_1} & \cdots & z_m^{i_1} \\ \vdots & \vdots & & \vdots \\ z_1^{i_N} & z_2^{i_N} & \cdots & z_m^{i_N} \end{pmatrix}. \tag{6.1}$$

In Matlab® syntax, the above matrix can be implemented as

$$V_{I,m}(z_1, \dots, z_m) = [\mathrm{ones}(N,1) * (z_1, \dots, z_m)].\wedge[(i_1, \dots, i_N)' * \mathrm{ones}(1,m)], \tag{6.2}$$

where $\mathbf{z}'$ denotes the transpose of vector $\mathbf{z}$.

Equal roots. In this case matrix $\mathcal{V}_{n,m}(z)$ in (3.19) can be replaced by the matrix $\mathcal{V}_{I,m}(z_1, \dots, z_m)$ defined for each set $I = \{i_1, \dots, i_N\}$ as

$$\mathcal{V}_{I,m}(z) = \begin{pmatrix} z^{i_1} & i_1 z^{i_1} & \cdots & i_1^{m-1} z^{i_1} \\ z^{i_2} & i_2 z^{i_2} & \cdots & i_2^{m-1} z^{i_2} \\ \vdots & \vdots & & \vdots \\ z^{i_N} & i_N z^{i_N} & \cdots & i_N^{m-1} z^{i_N} \end{pmatrix}. \tag{6.3}$$

The above matrix can be written as

$$\mathcal{V}_{I,m}(z) = \begin{pmatrix} 1 & i_1 & \cdots & i_1^{m-1} \\ 1 & i_2 & \cdots & i_2^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & i_N & \cdots & i_N^{m-1} \end{pmatrix} .* \begin{pmatrix} z^{i_1} & z^{i_1} & \cdots & z^{i_1} \\ z^{i_2} & z^{i_2} & \cdots & z^{i_2} \\ \vdots & \vdots & & \vdots \\ z^{i_N} & z^{i_N} & \cdots & z^{i_N} \end{pmatrix} \tag{6.4}$$

where $.*$ denotes the element-by-element matrix product implemented in Matlab®.

In this case the two matrices in (6.4) can be implemented as

$$\mathcal{V}^1_{I,m}(z) = [(i_1, \dots, i_N)' * \mathrm{ones}(1,m)].\wedge[\mathrm{ones}(N,1) * (0, 1, \dots, m-1)],$$
$$\mathcal{V}^2_{I,m}(z) = \left[\left(\mathrm{ones}(N,1) * z\right).\wedge(i_1, \dots, i_N)' \right] * \mathrm{ones}(1,m)],$$

where $.\wedge$ denotes the element-by-element power function implemented in Matlab®.

Distinct roots $z_1, \dots, z_m$ of higher multiplicities $d_1, \dots, d_m$. In general, for an ordered set of indices $I = \{i_1, \dots, i_N\}$ one can directly obtain the terms w_n, $n \in I$ of the sequence by considering the matrix

$$\mathcal{W}_{I,d_1,\dots,d_m}(z_1, \dots, z_m) = \left(\ \mathcal{V}_{I,d_1}(z_1) \ \middle| \ \cdots \ \middle| \ \mathcal{V}_{I,d_m}(z_m) \ \right), \tag{6.5}$$

where the matrix components $\mathcal{V}_{I,d_i}(z_i)$, $i = 1, \dots, m$ are defined in (6.3).

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